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MINIMAL QUASI-F COVERS OF υX

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Abstract. We show that if X is a space such that $\beta QF(X) = QF(\beta X)$ and each stable $Z(X)^{\#}$-ultrafilter has the countable intersection property, then there is a homeomorphism $h_X : \upsilon QF(X) \longrightarrow QF(\upsilon X)$ with $r_X = \Phi_{\upsilon X} \circ h_X$. Moreover, if $\beta QF(X) = QF(\beta X)$ and $\upsilon E(X) = E(\upsilon X)$ or $\upsilon\Lambda(X) = \Lambda(\upsilon X)$, then $\upsilon QF(X) = QF(\upsilon X)$.

1. Introduction

All spaces in this paper are Tychonoff spaces and $\beta X(\upsilon X$, resp.) denotes the Stone-Čech compactification(Hewitt realcompactification, resp.) of a space X .

Gleason([3]) and Iliadis([6]) characterized minimal extremally disconnected covers of spaces and the projective objects in the category of compact spaces and continuous maps and proved that each compact space has the projective cover which is the extremally diconnected cover of the space. Over the next decade numorous authors extended this characterization to other categories of topological spaces([9]).

A space is called *a quasi-F space* if its dense cozero-set is C^*-embedded. A quasi-F space, introduced by Henriksen and Gillman, is a generalization of F-spaces([2]), in which every cozero-set is C^*-embedded. Each space has the minimal quasi-F cover $(QF(X), \Phi_X)$([1], [4], [5]). In [5] and [8], authors investigated when $\beta QF(X) = QF(\beta X)$ and $QF(X) = \Phi_\beta^{-1}(X)$, where $(QF(\beta X), \Phi_\beta)$ is the minimal quasi-F cover of βX.

It is well-known that each space has the minimal extremally disconnected cover $(E(X), k_X)$ and that $\beta E(X) = E(\beta X)$([9]). Moreover, internal characterizations of a space X that is equivalent to $E(\upsilon X) =$

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$\upsilon E(X)$ is known([9]). Similar results for the minimal basically disconnected cover $(\Lambda X, \Lambda_X)$ are given by [8].

In this paper, for any space X such that $\beta QF(X) = QF(\beta X)$ and each stable $Z(X)^{\#}$-ultrafilter has the countable intersection property, we first show that there is a map $r_X : \upsilon QF(X) \longrightarrow \upsilon X$ such that $(\upsilon QF(X), r_X)$ is a quasi-F cover of υX and show that $\upsilon QF(X) = QF(\upsilon X)$, that is, there is a homeomorphism $h_X : \upsilon QF(X) \longrightarrow QF(\upsilon X)$ with $r_X = \Phi_{\upsilon X} \circ h_X$. Moreover, if $\beta QF(X) = QF(\beta X)$ and $\upsilon E(X) = E(\upsilon X)$ or $\upsilon\Lambda(X) = \Lambda(\upsilon X)$, then $\upsilon QF(X) = QF(\upsilon X)$.

For the terminology, we refer to [2, 9].

2. Quasi-F covers

Let X be a space. It is well-known that the collection $\mathcal{R}(X)$ of all regular closed sets in X, when partially ordered by inclusion, becomes a complete Boolean algebra, in which the join, meet, and complementation operations are defined as follows :
For any $A \in \mathcal{R}(X)$ and any $\mathcal{F} \subseteq \mathcal{R}(X)$,
$\bigvee \mathcal{F} = cl_X(\cup\{F \mid F \in \mathcal{F}\})$,
$\bigwedge \mathcal{F} = cl_X(int_X(\cap\{F \mid F \in \mathcal{F}\}))$, and
$A' = cl_X(X - A)$.
A sublattice of $\mathcal{R}(X)$ is a subset of $\mathcal{R}(X)$ that contains $\emptyset$, X and is closed under finite joins and finite meets([9]).

A map $f : Y \longrightarrow X$ is called *a covering map* if it is an onto continuous, perfect, and irreducible map([9]).

Lemma 2.1. ([8])

(1) *Let X be a dense subspace of Y. Then the map $\phi : R(Y) \longrightarrow R(X)$, defined by $\phi(A) = A \cap X$, is a Boolean isomorphism.*
(2) *Let $f : Y \longrightarrow X$ be a covering map. Then the map $\psi : R(Y) \longrightarrow R(X)$, defined by $\psi(A) = f(A)$, is a Boolean isomorphism.*

In the above lemma, the inverse map $\phi^{-1} : R(X) \longrightarrow R(Y)$ of ϕ is given by $\phi^{-1}(B) = cl_Y(B)$ $(B \in R(X))$ and the inverse map $\psi^{-1} : R(X) \longrightarrow R(Y)$ of ψ is given by $\psi^{-1}(B) = cl_Y(int_Y(f^{-1}(B))) = cl_Y(f^{-1}(int_X(B)))$ $(B \in R(X))$.

Lemma 2.2. *Let $f : Y \longrightarrow X$ be a covering map and (A_n) a decreasing sequence in $R(Y)$. Then $f(\cap\{A_n \mid n \in N\}) = \cap\{f(A_n) \mid n \in N\}$.*

Proof. Let (A_n) be a decreasing sequence in $R(Y)$. Then clearly, $f(\cap\{A_n \mid n \in N\}) \subseteq \cap\{f(A_n) \mid n \in N\}$.

Let $x \in \cap\{f(A_n) \mid n \in N\}$ and $n \in N$. Then $x \in f(A_n)$ and $A_n \cap f^{-1}(x) \neq \emptyset$ for all $n \in N$. Since (A_n) is a decreasing sequence in $R(Y)$, (A_n) has the finite intersection property and $\{A_n \cap f^{-1}(x) \mid n \in N\}$ is a family of closed sets in $f^{-1}(x)$ with the finite intersection property. Since f is a compact map, $f^{-1}(x)$ is a compact subset of Y. Hence $\cap\{A_n \cap f^{-1}(x) \mid n \in N\} \neq \emptyset$. Pick $y \in \cap\{A_n \cap f^{-1}(x) \mid n \in N\}$. Then $y \in \cap\{A_n \mid n \in N\}$ and $f(y) = x$ and $\cap\{f(A_n) \mid n \in N\} \subseteq f(\cap\{A_n \mid n \in N\})$. $\square$

Definition 2.3. A space X is called *a quasi-F space* if for any zero-sets A, B in X, $cl_X(int_X(A \cap B)) = cl_X(int_X(A)) \cap cl_X(int_X(B))$, equivalently, every dense cozero-set in X is C^*-embedded in X.

It is well-known that a space X is a quasi-F space if and only if βX(or υX) is a quasi-F space.

Definition 2.4. Let X be a space. Then a pair (Y, f) is called

(1) *a cover of* X if $f : X \longrightarrow Y$ is a covering map,
(2) *a quasi-F cover of* X if (Y, f) is a cover of X and Y is a quasi-F space, and
(3) *a minimal quasi-F cover of* X if (Y, f) is a quasi-F cover of X and for any quasi-F cover (Z, g) of X, there is a covering map $h : Z \longrightarrow Y$ such that $f \circ h = g$.

Let X be a space, $Z(X) = \{Z \mid Z \text{ is a zero-set in } X\}$ and $Z(X)^{\#} = \{cl_X(int_X(A)) \mid A \in Z(X)\}$. Then $Z(X)^{\#}$ is a sublattice of $R(X)$.

Suppose that X is a compact space. Let $QF(X) = \{\alpha \mid \alpha \text{ is a } Z(X)^{\#}\text{-ultrafilter}\}$ and for any $A \in Z(X)^{\#}$, let $\Sigma_A = \{\alpha \in QF(X) \mid A \in \alpha\}$. Then the space $QF(X)$, equipped with the topology for which $\{QF(X) - \Sigma_A \mid A \in Z(X)^{\#}\}$ is a base, is a quasi-F space. Define the map $\Phi_X : QF(X) \longrightarrow X$ by $\Phi_X(\alpha) = \cap\{A \mid A \in \alpha\}$. Then $(QF(X), \Phi_X)$ is the minimal quasi-F cover of X and for any $A \in Z(X)^{\#}$, $\Phi_X(\Sigma_A) = A$([4]).

We recall that a space X is called *a weakly Lindelöf space* if for any open cover $\mathcal{U}$ of X, there is a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\cup\{V \mid V \in \mathcal{V}\}$ is a dense subset of X and that X is called *a locally weakly Lindelöf space* if every element of X has a weakly Lindelöf neighborhood.

For any weakly Lindelöf space, $\beta QF(X) = QF(\beta X)$, that is, there is a homeomorphism $h : \beta QF(X) \longrightarrow QF(\beta X)$ such that $\beta_X \circ \Phi_X = \Phi_{\beta X} \circ h \circ \beta_{QF(X)}$([5]). And for any locally weakly Lindelöf space, $QF(X)$ is the subspace of $\{\alpha \in QF(\beta X) \mid \cap\{A \mid A \in \alpha\} \in X\}$ of $Q(\beta X)$([8]).

Since $Z(X)^{\#}$ and $Z(\beta X)^{\#}$ are isomorphic, $QF(\beta X)$ is homeomorphic to the space $\{\alpha \mid \alpha \text{ is a } Z(X)^{\#}\text{-ultrafilter}\}$ which is equipped with the topology for which $\{\Sigma'_A \mid A \in Z(X)^{\#}\}$ is a base for closed sets, where $\Sigma'_A = \{\alpha \mid \alpha$ is a $Z(X)^{\#}$-ultrafilter and $A \in \alpha\}$.

Let X be a space and $(QF(\beta X), \Phi_\beta)((QF(\upsilon X), \Phi_\upsilon)$, resp.) denote the minimal quasi-F cover of $\beta X(\upsilon X$, resp.).

Definition 2.5. Let X be a space. Then a $Z(X)^{\#}$-filter α is called *fixed* if $\cap\{A \mid A \in \alpha\} \neq \emptyset$.

Suppose that $QF(X) = \Phi_\beta^{-1}(X)$, where $\Phi_\beta^{-1}(X)$ is the subspace of $QF(\beta X)$. Then $QF(X) = \{\alpha \mid \alpha$ is a fixed $Z(X)^{\#}$-ultrafilter$\}$ and $\{\sigma_A \mid A \in Z(X)^{\#}\}$ is a base for closed sets of $QF(X)$, where $\sigma_A = \{\alpha \mid \alpha$ is a fixed $Z(X)^{\#}$-filter and $A \in \alpha\}$.

Lemma 2.6. ([8]) *Let X be a space. Then $\Phi_\beta^{-1}(X)$ is a quasi-F space if and only if $QF(X) = \Phi_\beta^{-1}(X)$ and $\Phi_X = \Phi_{\beta_X}$, that is, $\Phi_X(\alpha) = \cap\{A \mid A \in \alpha\}$, where $\Phi_{\beta_X} : \Phi_\beta^{-1}(X) \longrightarrow X$ is the restriction and corestriction of Φ_β with respect to $\Phi_\beta^{-1}(X)$ and X, respectively.*

Let X be a space. Then there is a continuous map $\Phi^\beta : \beta QF(X) \longrightarrow \beta X$ such that $\beta_X \circ \Phi_X = \Phi^\beta \circ \beta_{QF(X)}$. Since $\beta QF(X)$ and βX are compact spaces and $\beta_X : X \longrightarrow \beta X$ and $\beta_{QF(X)} : QF(X) \longrightarrow \beta QF(X)$ are dense embeddings, Φ^β is a covering map. Hence $(\beta QF(X), \Phi^\beta)$ is a quasi-F cover of βX and there is a covering map $q_X : \beta QF(X) \longrightarrow QF(\beta X)$ such that $\Phi^\beta = \Phi_\beta \circ q_X$.

For any space X and $x \in X$, let $\delta(x) = \{A \in Z(X)^{\#} \mid x \in int_X(A)\}$.

Proposition 2.7. *Let X be a space such that $\Phi_\beta^{-1}(X)$ is a quasi-F space and α a fixed $Z(X)^{\#}$-ultrafilter. Then we have the following :*

(1) *$\Phi_X(\alpha) = x$ if and only if $\delta(x) \subseteq \alpha$,*
(2) *for any $A \in Z(X)^{\#}$, $\Phi_X(\sigma_A) = A$, and*
(3) *for any decreasing sequence (A_n) in $Z(X)^{\#}$, $\Phi_X(\cap\{\sigma_{A_n} \mid n \in N\}) = \cap\{A_n \mid n \in N\}$.*

Proof. (1) ($\Rightarrow$) Let $\Phi_X(\alpha) = x$. Suppose that there is an $A \in \delta(x) - \alpha$. Since $A \notin \alpha$ and α is a $Z(X)^{\#}$-ultrafilter, $A' \in \alpha$ and $\Phi_X(\alpha) = x \in int_X(A) = X - A'$. This is a contradiction. Hence $\delta(x) \subseteq \alpha$.

($\Leftarrow$) Suppose that $\delta(x) \subseteq \alpha$. Note that $\delta(x)$ is a local base at x in X. Let $A \in \alpha$. Then for any $B \in \delta(x)$, $B \in \alpha$ and $A \cap B \neq \emptyset$. Hence $x \in cl_X(A) = A$ and so $x \in \cap\{A \mid A \in \alpha\} = \Phi_X(\alpha)$.

(2) Let $A \in Z(X)^{\#}$ and $\alpha \in \sigma_A$. Then $A \in \alpha$ and $\Phi_X(\alpha) = \cap\{B \mid B \in \alpha\} \in A$. Hence $\Phi_X(\sigma_A) \subseteq A$.

Suppose that $y \in A$. Then $\delta(y) \cup \{A\}$ has the finite meet property such that $\delta(y) \cup \{A\} \subseteq Z(X)^{\#}$. By Zorn's lemma, there is a $Z(X)^{\#}$-ultrafilter γ such that $\delta(y) \cup \{A\} \subseteq \gamma$. By (1), $\Phi_X(\gamma) = y$ and since $A \in \gamma$, $\gamma \in \sigma_A$. Hence $y \in \Phi_X(\sigma_A)$ and $A \subseteq \Phi_X(\sigma_A)$.

(3) By Lemma 2.2, it is trivial. □

3. Minimal quasi-F covers of υX

Iliadis([6])(Vermeer([8]), resp.) showed that every space X has the minimal extremally disconnected cover $(E(X), k_X)$(the minimal basically disconnected cover $(\Lambda(X), \Lambda_X)$, resp.) of X.

We recall that an $\mathcal{A}$-ultrafilter α is called *stable* if $\cap\{cl_{\beta X}(A) \mid A \in \alpha\} \subseteq \upsilon X$, where $\mathcal{A}$ is a sublattice of $R(X)$.

In [9] and [7], we can find internal characterizations of a space X which $E(\upsilon X) = \upsilon E(X)$ and $\Lambda(\upsilon X) = \upsilon\Lambda(X)$. In fact, for any space X(for any space X with $\beta\Lambda(X) = \Lambda(\beta X)$, resp.), the following are equivalent :

(1) $E(\upsilon X) = \upsilon E(X)(\Lambda(\upsilon X) = \upsilon\Lambda(X)$, resp.),
(2) for any decreasing sequence (A_n) in $R(X)(\sigma Z(X)^{\#}$, resp.) with $\cap\{A_n \mid n \in N\} = \emptyset$, $\cap\{cl_{\upsilon X}(A_n) \mid n \in N\} = \emptyset$,
(3) for any decreasing sequence (A_n) in $R(X)(\sigma Z(X)^{\#}$, resp.), $cl_{\upsilon X}(\cap\{A_n \mid n \in N\}) = \cap\{cl_{\upsilon X}(A_n) \mid n \in N\}$, and
(4) every stable $R(X)$-ultrafilter($\sigma Z(X)^{\#}$-ultrafilter, resp.) has the countable intersection property

where $\sigma Z(X)^{\#}$ is the smallest complete Boolean subalgebra of $R(X)$ such that it is closed under countable meets and $Z(X)^{\#} \subseteq \sigma Z(X)^{\#}$.

Definition 3.1. (1) A covering map $f : Y \longrightarrow X$ is called *$z^{\#}$-irreducible* if $f(Z(Y)^{\#}) = Z(X)^{\#}$.
(2) A subspace X of a space Y is called *$z^{\#}$-embedded in* Y if for any $A \in Z(X)^{\#}$, there is a $B \in Z(Y)^{\#}$ such that $A = B \cap X$.

It is well-known that for any compact space X, $\Phi_X : QF(X) \longrightarrow X$ is $z^{\#}$-irreducible([4]) and that every C^*-embedded subspace W of a space Z is $z^{\#}$-embedded in Z([2]).

Let $f : Y \longrightarrow X$ be a covering map. Since $Z(X)^{\#} \subseteq f(Z(Y)^{\#})$, $f : Y \longrightarrow X$ is $z^{\#}$-irreducible if and only if $f(Z(Y)^{\#}) \subseteq Z(X)^{\#}$. Using this, we have the following :

Proposition 3.2. *Let $f : Y \longrightarrow X$ and $g : X \longrightarrow W$ be covering maps. Then $g \circ f$ is $z^{\#}$-irreducible if and only if $f : Y \longrightarrow X$ and $g : X \longrightarrow W$ are $z^{\#}$-irreducible.*

Let X be a space. Since υX is a realcompact space, there is a continuous map $r_X : \upsilon QF(X) \longrightarrow \upsilon X$ such that $\upsilon_X \circ \Phi_X = r_X \circ \upsilon_{QF(X)}$.

Theorem 3.3. *Let X be a space such that $\beta QF(X) = QF(\beta X)$. Suppose that every stable $Z(X)^{\#}$-ultrafilter has the countable intersection property. Then we have the following :*

(1) *for any $x \in \upsilon X$, $\Phi_\beta^{-1}(x) = r_X^{-1}(x)$, and*

(2) *$r_X : \upsilon QF(X) \longrightarrow \upsilon X$ is a covering map.*

Proof. (1) Since $\beta QF(X) = QF(\beta X)$, $\upsilon QF(X)$ is a C^*-embedded subspace of $QF(\beta X)$.

Let $x \in \upsilon X$. Since Φ_β is an onto map, $\Phi_\beta^{-1}(x) \neq \emptyset$. Suppose that there is an $\alpha \in \Phi_\beta^{-1}(x) - \upsilon QF(X)$. Then there is a zero-set Z in $\beta QF(X)$ such that $\alpha \in Z$ and $Z \cap \upsilon QF(X) = \emptyset$([2]) and there is a real-valued continuous map f on X such that $f^{-1}(0) = Z$. For any $n \in N$, let $Z_n = cl_{\beta QF(X)}(int_{\beta QF(X)}(f^{-1}([\frac{1}{n}, \frac{1}{n}])))$. Then (Z_n) is a decreasing sequence in $Z(\beta QF(X))^{\#}$ such that $Z = \cap\{Z_n \mid n \in N\}$ and $\alpha \in int_{\beta QF(X)}(Z_n)$ for all $n \in N$. Moreover, $\cap\{Z_n \mid n \in N\} \cap QF(X) = \emptyset$. Since Φ_β is $z^{\#}$-irreducible, $(\Phi_\beta(Z_n))$ is a decreasing sequence in $Z(\beta X)$. By Lemma 2.1, $\alpha_X = \{A \cap X \mid A \in \alpha\}$ is a $Z(X)^{\#}$-ultrafilter.

Let $n \in N$. Since Φ_β is $z^{\#}$-irreducible, there is an $A_n \in Z(\beta X)$ such that $\alpha \in \sigma_{A_n} = Z_n$. Hence $\Phi_\beta(\alpha) \in A_n = \Phi_\beta(Z_n)$. Since $A_n \in \alpha$, $\Phi_\beta(Z_n) \in \alpha$ and $\Phi_\beta(Z_n) \cap X \in Z(X)^{\#}$. Note that $x \in \cap\{cl_{\beta X}(A) \mid A \in \alpha\} = \cap\{cl_{\beta X}(B) \mid B \in \alpha_X\}$. Since $x \in \upsilon X$, α_X is a stable $Z(X)^{\#}$-ultrafilter and by the assumption, α_X has the countable intersection property. Thus $\cap\{\Phi_\beta(Z_n) \cap X \mid n \in N\} \neq \emptyset$. Since $\beta QF(X) = QF(\beta X)$, $QF(X) = \Phi_\beta^{-1}(X)$ and $\{Z_n \cap \Phi_\beta^{-1}(x) \mid n \in N\}$ is a family of closed sets in $\Phi_\beta^{-1}(x)$ with the finite intersection property. Since $\Phi_\beta^{-1}(x)$ is compact, $\cap\{Z_n \mid n \in N\} \neq \emptyset$. This is a contradiction and so $r_X^{-1}(x) = \Phi_\beta^{-1}(x) \subseteq \upsilon QF(X)$.

(2) By (1), $r_X : \upsilon \Lambda X \longrightarrow \upsilon X$ is an onto, compact map.

Let F be a closed set in $\upsilon QF(X)$ and $x \in \upsilon X - r_X(F)$. Then $r_X^{-1}(x) \cap F = \emptyset$ and since $r_X^{-1}(x)$ is a compact subset of $\upsilon QF(X)$ and $\upsilon QF(X)$ is a subspace of $QF(\beta X)$, there is an $A \in Z(\beta X)^{\#}$ such that $r_X^{-1}(x) \cap \Sigma_A = \emptyset$ and $F \subseteq \Sigma_A$, because $\{\Sigma_A \mid A \in Z(\beta X)^{\#}\}$ is a base for closed sets in $\beta QF(X) = QF(\beta X)$. Since $\Phi_\beta(\Sigma_A) = A$, $\Phi_\beta(F) = r_X(F) \subseteq A$. Then

$x \notin A$ and since A is a closed set in βX, $cl_{\upsilon X}(r_X(F)) \subseteq A \cap \upsilon X$. Hence $x \notin cl_{\upsilon X}(r_X(F))$ and $cl_{\upsilon X}(r_X(F)) \subseteq r_X(F)$. Thus r_X is a closed map and r_X is a covering map. $\square$

PROPOSITION 3.4. *Let X be a space. Then the following are equivalent :*

(1) *for any decreasing sequence (A_n) in $Z(X)^{\#}$ with $\cap\{A_n \mid n \in N\} = \emptyset$, $\cap\{cl_{\upsilon X}(A_n) \mid n \in N\} = \emptyset$,*
(2) *for any decreasing sequence (A_n) in $Z(X)^{\#}$, $cl_{\upsilon X}(\cap\{A_n \mid n \in N\}) = \cap\{cl_{\upsilon X}(A_n) \mid n \in N\}$, and,*
(3) *every stable $Z(X)^{\#}$-ultrafilter has the countable intersection property.*

Proof. (1) $\Rightarrow$ (2) Clearly, $cl_{\upsilon X}(\cap\{A_n \mid n \in N\}) \subseteq \cap\{cl_{\upsilon X}(A_n) \mid n \in N\}$. Let $x \in \cap\{cl_{\upsilon X}(A_n) \mid n \in N\} - cl_{\upsilon X}(\cap\{A_n \mid n \in N\})$. Then there is a $B \in Z(\upsilon X)^{\#}$ such that $x \in int_{\upsilon X}(B)$ and $B \cap (\cap\{A_n \mid n \in N\}) = \emptyset$. Let $C = B \cap X$. Then by Lemma 2.1, $C \in Z(X)^{\#}$ and $\{C \wedge A_n \mid n \in N\}$ is a decreasing sequence in $Z(X)^{\#}$ with $\cap\{C \wedge A_n \mid n \in N\} = \emptyset$. By (1), $\cap\{cl_{\upsilon X}(C \wedge A_n) \mid n \in N\} = \emptyset$.

Let U be a neighborhood of x in υX and $n \in N$. Then $U \cap int_{\upsilon X}(B) \cap A_n \neq \emptyset$. Note that

$$\begin{aligned}
& int_{\upsilon X}(B) \cap A_n \\
&= (int_{\upsilon X}(B) \cap X) \cap A_n \\
&\subseteq int_X(B \cap X) \cap A_n \\
&= int_X(C) \cap A_n \\
&\subseteq cl_X(int_X(C) \cap int_X(A_n)) \\
&= C \wedge A_n.
\end{aligned}$$

Since $U \cap int_{\upsilon X}(B) \cap A_n \neq \emptyset$, $U \cap (C \wedge A_n) \neq \emptyset$ and $x \in C \wedge A_n$. Hence $x \in \cap\{cl_{\upsilon X}(C \wedge A_n) \mid n \in N\}$. This is a contradiction. Thus $\cap\{cl_{\upsilon X}(A_n) \mid n \in N\} \subseteq cl_{\upsilon X}(\cap\{A_n \mid n \in N\})$.

(2) $\Rightarrow$ (3) Let α be a stable $Z(X)^{\#}$-ultrafilter and (A_n) a sequence in α. For any $n \in N$, $B_n = \wedge\{A_i \mid i = 1, 2, \cdots, n\}$. Then (B_n) is a decreasing sequence in $Z(X)^{\#}$. Since α is stable, there is an $x \in \upsilon X$ such that $x \in \cap\{cl_{\beta X}(A_n) \mid n \in N\}$ and so $x \in \cap\{cl_{\upsilon X}(A_n) \mid n \in N\}$. By (2), $x \in cl_{\upsilon X}(\cap\{A_n \mid n \in N\})$ and $\cap\{A_n \mid n \in N\} \neq \emptyset$. Hence α has the countable intersection property.

(3) $\Rightarrow$ (1) Let (A_n) be a decreasing sequence in $Z(X)^{\#}$ such that $\cap\{A_n \mid n \in N\} = \emptyset$. Suppse that $\cap\{cl_{\upsilon X}(A_n) \mid n \in N\} \neq \emptyset$. Pick $x \in \cap\{cl_{\upsilon X}(A_n) \mid n \in N\}$. Let $\alpha_0 = \{B \cap X \mid B \in Z(\upsilon X)^{\#}, x \in$

$int_{\upsilon X}(B)\} \cup \{A_n \mid n \in N\}$. Then clearly, α_0 is a subset of $Z(X)^{\#}$ with the finite meet property and by Zorn's lemma, there is a $Z(X)^{\#}$-ultrafilter α such that $\alpha_0 \subseteq \alpha$. Since $\{B \in Z(\upsilon X)^{\#} \mid x \in int_{\upsilon X}(B)\}$ is a local base at x in υX, $x \in \cap\{cl_{\upsilon X}(A) \mid A \in \alpha\}$ and α is a stable $Z(X)^{\#}$-ultrafilter. By (3), α has the countable intersection property and since $\{A_n \mid n \in N\} \subseteq \alpha$, $\cap\{A_n \mid n \in N\} \neq \emptyset$. It is a contradiction and thus $\cap\{cl_{\upsilon X}(A_n) \mid n \in N\} = \emptyset$. $\square$

Let X be a space with $QF(\beta X) = \beta QF(X)$. Then $\Phi_X : QF(X) \longrightarrow X$ is $z^{\#}$-irreducible([5]).

Theorem 3.5. *Let X be a space with $QF(\beta X) = \beta QF(X)$. Suppose that every stable $Z(X)^{\#}$-ultrafilter has the countable intersection property. Then $QF(\upsilon X) = \upsilon QF(X)$, that is, there is a homeomorphism $h_X : \upsilon QF(X) \longrightarrow QF(\upsilon X)$ such that $r_X = \Phi_\upsilon \circ h_X$.*

Proof. By Theorem 3.3, $r_X : \upsilon\Lambda X \longrightarrow \upsilon X$ is a covering map. Since $(\upsilon QF(X), r_X)$ is a quasi-F cover of υX, there is a covering map $h_X : \upsilon QF(X) \longrightarrow QF(\upsilon X)$ such that $r_X = \Phi_\upsilon \circ h_X$. Since $QF(\beta X) = \beta QF(X)$, Φ_X is $z^{\#}$-irreducible. Note that $r_X \circ \upsilon_{QF(X)} = \upsilon_X \circ \Phi_X$ and $\upsilon_{QF(X)}$, υ_X are C^*-embedded in $\upsilon QF(X)$ and υX, respectively. Hence r_X is $z^{\#}$-irreducible and by Proposition 3.2, h_X is $z^{\#}$-irreducible.

Let $p \neq q$ in $\upsilon QF(X)$. Then there are A, B in $Z(\upsilon QF(X))^{\#}$ such that $p \in A$, $q \in B$ and $A \cap B = \emptyset$. Hence $A \wedge B = \emptyset$ and by Lemma 2.1, $\emptyset = h_X(A \wedge B) = h_X(A) \wedge h_X(B)$. Since h_X is $z^{\#}$-irreducible, $h_X(A) \in Z(QF(\upsilon X))^{\#}$ and $h_X(B) \in Z(QF(\upsilon X))^{\#}$. Since $QF(\upsilon X)$ is a quasi-F space, $h_X(A) \wedge h_X(B) = h_X(A) \cap h_X(B) = \emptyset$. Since $h_X(p) \in h_X(A)$ and $h_X(q) \in h_X(B)$, $h_X(p) \neq h_X(q)$ and h_X is one-to-one. Hence $\upsilon QF(X) = QF(\upsilon X)$. $\square$

For any space X, $Z(X)^{\#} \subseteq \sigma Z(X)^{\#} \subseteq R(X)$. By Theorem 3.3, Proposition 3.4, and Theorem 3.5, we have the following :

Corollary 3.6. *Let X be a space such that $QF(\beta X) = \beta QF(X)$. If $E(\upsilon X) = \upsilon E(X)$ or $\Lambda(\upsilon X) = \upsilon\Lambda(X)$, then $QF(\upsilon X) = \upsilon QF(X)$.*

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